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EXPLICIT SOLUTION OF A DELAYED INVESTMENT PROBLEM

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ABSTRACT

We present an explicit solution to an optimal stopping (investment) problem in a modal described by stochastic delay differential equation representing the dynamics of a risky assets (for example stocks). The method of finding the required explicit solution is expressed in terms of the solution of a corresponding free boundary value problem.

***Keywords**: Stochastic delay differential equations, Optimal stopping, Free-Boundary problem, Itô's formula.*

I. INTRODUCTION

Suppose that a person's stock in a given market varies according to the following 2- dimensional stochastic delay differential equation (SDDE):

$$dX_1(t) = \left[\theta_1 X_1(t) + \alpha_1 \int_{-\delta}^{0} \exp(\lambda s) X_1(t+s)\, ds + \beta_1 X_1(t-s)\right] dt$$

$$+ \sigma_1 X_1(t) + \beta_1 \exp\left[(\lambda\delta) \int_{-\delta}^{0} \exp(\lambda s) X_1(t+s)\, ds\right] dB_1,\ t \geq 0 \tag{1,1}$$

$$X_1(s) = \xi(s) \quad for\ -\delta \leq s \leq 0 \tag{1.2}$$

$$dX_2(t) = \left[\theta_2 X_2(t) + \alpha_2 \int_{-\delta}^{0} \exp(\lambda s) X_2(t+s)\, ds + \beta_2 X_2(t+s)\right] dt$$

$$+ \sigma_2 X_2(t) + \beta_2 \exp\left[(\lambda\delta) \int_{-\delta}^{0} \exp(\lambda s) X_2(t+s)\, ds\right] dB_2,\ t \geq 0 \tag{1.3}$$

$$X_2(s) = \eta(s) \quad for\ -\delta \leq s \leq 0 \tag{1.4}$$

Where $b_i : R^3 \to R$ and $\sigma_i : R^3 \to R$ are given functions , $(i = 1,2)$, δ is the (constant) delay, λ R is a constant and $B(t) = B_1(t,\varpi), B_2(t,\varpi) \quad t \geq 0,\ \varpi \in \Omega$ is a 2-dimensional Brownian motion.

The solution of (1.1) , (1.3) with initial path (1.2) , (1.4) is denoted by $\left(X_1^{\xi}, X_2^{\eta}\right)(t)$. For conditions for existence and uniqueness of solutions for such equations see [M1], [M2]. The law of $\left(X_1^{\xi}, X_2^{\eta}\right)(t)$ is denoted by a $Q^{\varsigma\eta}(t)$ and the corresponding expectation by $E^{\varsigma\eta}$.

Historically the optimal stopping problem was strictly connected with the problem of the optimal exercise time for the American put option. It has a deep connection with free-boundary problems.
Since the dynamics of the considered system is a delay system, it is known that the studies of such problem are quite hard. The difficulty arises from the fact that it has an infinite dimensional nature.
In our approach to solve the given optimal stopping problem, we assume that the state variable to be composed of a couple. The first component in this couple is a real variable carrying the present of the system while the second one is some weighted average. This assumption reduces our infinite dimensional problem to a finite dimensional one.

II. THE OPTIMAL STOPPING PROBLEM

Let $X_t(s) = X(t+s)$, $-\delta \le s \le 0$, $t \ge 0$. i.e. X_t is the segment of the path of X_t from $t-\delta$ to t.

Let $X(t) = \left(X_1^{\xi}, X_2^{\eta}\right)(t)$ be the solution of the systems (1.1) - (1.3) and let g (the reward function) be a given function on R^2 satisfying the following conditions:

(i) $g(\xi,\eta) \ge 0$, $\forall (\xi,\eta) \in R^2$

(ii) g is continuous

Find the optimal expected reward and the corresponding stopping time t^* for $X(t)$ such that:

$$\Phi\left(s(\xi,\eta)\right) = \sup E^{s(\xi,\eta)}\left[g\left(\tau,\left(X_1^{\xi}(\tau), Y_1\right),\left(X_2^{\eta}(\tau), Y_2\right)\right)\right] \qquad (2.1)$$

Where

$$Y_1(t) = \int_{-\delta}^{0} \exp(\lambda s)\, X_1(t+s)\, ds \qquad (2.2)$$

and

$$Y_2(t) = \int_{-\delta}^{0} \exp(\lambda s)\, X_2(t+s)\, ds \qquad (2.3)$$

We assume that the value function Φ depends on the initial path (ξ, η) only through the following four linear functionals:

$$X_1 = X_1(\zeta) = \xi(0) \qquad (2.4)$$

$$X_2 = X_2(\eta) = \eta(0) \qquad (2.5)$$

$$Y_1 = Y_1(\xi) = \int_{-\delta}^{0} \exp(\lambda s)\, \xi(s)\, ds \qquad (2.6)$$

$$Y_2 = Y_2(\eta) = \int_{-\delta}^{0} \exp(\lambda s)\, \eta(s)\, ds \qquad (2.7)$$

$$\Phi(\xi,\eta) = \Phi\left(X_1, X_2, Y_1, Y_2\right) \qquad (2.8)$$

Where $\Phi : R^4 \to R$ Such idea is used in [EOS] .

III. VARIATIONAL IN EQALITY FORMULATION

Suppose the functional $F : R \times R \times C[-\delta, 0] \times C[-\delta, 0] \to R$ is of the form:

$F\left(t, X_1, X_{2,}, \eta_1, \eta_2\right) = F\left(t, X_1, X_{2,}, Y_1(\eta_1), Y_2(\eta_2)\right)$ for some function $f \in C^{1,2,2,1,1}\left(R^3\right)$.

Ito formula

Define

$$G(t) = F\left(s+t, \left(X_1^{\xi}(t), Y_1\left(X_1^{\xi}(.)\right)\right). X_1^{\eta}(t), Y_2\left(X_1^{\eta}(.)\right)\right) \tag{3.1}$$

Then we have

$$d\,G(t) = L f\, dt + \frac{\partial f}{\partial x_1}\sigma_1\left(x_1, y_1, z_1\right) d B_1(t) + \frac{\partial f}{\partial x_2}\sigma_2\left(x_2, y_2, z_2\right) d B_2(t)$$

$$+\frac{\partial f}{\partial y_1}\left(x_1 - \exp\left(-\lambda\delta z_1\right) - \lambda y_1\right) dt + \frac{\partial f}{\partial y_2}\left(x_2 - \exp\left(-\lambda\delta z_2\right) - \lambda y_2\right) dt + \tag{3.2}$$

Where

$$L\,F = L\,F\left(u, x_1, x_2, y_1, y_2, z_1, z_2\right) = \frac{\partial f}{\partial u} + b_1\left(x_1, y_1, z_1\right)\frac{\partial f}{\partial x_1} + b_2\left(x_2, y_2, z_2\right)\frac{\partial f}{\partial x_2}$$

$$\frac{1}{2}\sigma_1^2\left(x_1, y_1, z_1\right)\frac{\partial^2 f}{\partial x^2{}_1} + \frac{1}{2}\sigma_2^2\left(x_2, y_2, z_2\right)\frac{\partial^2 f}{\partial x^2{}_2} + 2\,\sigma_1\left(x_1, y_1, z_1\right)\sigma_2\left(x_2, y_2, z_2\right)\frac{\partial^2 f}{\partial x_1 \partial x_2} \tag{3.3}$$

And $L\,F\left(u, x_1, x_2, y_1, y_2, z_1, z_2\right)$ and the other function are evaluated at

$u = s+t$, $x_1 = x_1\left(X_1^{\xi}(.)\right) = X_1^{\xi}(t)$, $x_2 = x_2\left(X_t^{\xi}(.)\right) = X_2^{\eta}(t)$, $Y_1 = Y_1\left(X_t^{\xi}(.)\right)$

$$= \int_{-\delta}^{0} \exp(\lambda s) X_1^{\xi}(t+s)\, ds,\; Y_2 = Y_2\left(X_2^{\eta}(.)\right) = \int_{-\delta}^{0} \exp(\lambda s) X_2^{\eta}(t+s)\, ds \tag{3.4}$$

and

$z_2 = z_2\left(X_t^{\xi}(.)\right) = X_2^{\xi}(t-\delta)$, $z_2 = z_2\left(X_t^{\eta}(.)\right) = X_2^{\eta}(t-\delta)$.

Dynkin formula

Let $f \in C^{1,2,2,1,1}\left(R^5\right)$. Then for t 0 we have

$$E^{s,\xi,\eta}\left[f\left(t+s, X_1^{\xi}(t), y_1\left(X_t^{\xi}(.)\right), X_2^{\eta}(t), y_1\left(X_t^{\eta}(.)\right)\right)\right] = f\left(s, \xi(0), y_1(\xi), \eta(0), y_2(\eta)\right)$$

$$+ E^{s,\xi,\eta}\left[\int_0^t \left(L f + \frac{\partial f}{\partial y_1}\left[x_1 - (-\lambda\,\delta) z_1 - \lambda\, y_1\right] + \frac{\partial f}{\partial y_2}\left[x_2 - \exp(-\lambda\,\delta) z_2 - \lambda\, y_2\right]\right) dr\right] \tag{3.5}$$

Where $L\,F\left(u, x_1, x_2, y_1, y_2, z_1, z_2\right)$ and the other functions in the curly bracket are evaluated at

$u = s+t$, $x_1 = X_1^{\xi}(r)$, $x_2 = X_2^{\eta}(r)$, $y_1 = y_1(X_t^{\xi}(.))$, $y_2 = y_2(X_r^{\xi}(.))$,

$z_1 = X_1^{\xi}(r-\delta)$, $z_2 = X_2^{\eta}(r-\delta)$

Theorem 1 (Verification theorem). Let $S = \left(R^{+}\right)^5$.
Suppose we can find a nonnegative function φ: →R such that
(i) $\phi \in C^{1.2.2.1.1}\left(S^0\right) \cap \left(\overline{S}\right)$
(ii) $\phi\left(x_1, x_2, y_1, y_2\right) \geq g\left(x_1, x_2, y_1, y_2\right)$ on S and $=$ g on ∂S
Define the continuation region D. Assume D has the form:
(iii) $D := \{ \left(x_1, x_2, y_1, y_2\right) \in S, v_1\left(x_1, y_1\right) \leq \mu v_1\left(x_2, y_2\right) \}$, For some Lipschitz continuous function $u : R^2 \to [0, \infty)$
(iv) $\phi \in C^{1.2.2.1.1}\left(S \setminus \partial D\right)$, where the second order derivatives of with respect to x_1, x_2 are locally bounded near D
(v) $L\ \phi \leq 0$ on $S \setminus \overline{D}$
(vi) $L\ \phi = 0\ D$
(vii) The family $\left\{\phi\left(X_1^{\xi}(\tau), y_1\left(X_\tau^{\xi}(.)\right), X_2^{\eta}(\tau), y_2\left(X_\tau^{\eta}(.)\right)\right) ; \tau \leq \tau_D\right\}$
is uniformly integrable w. r. t. $Q^{\xi\,\eta}\ \forall\ (\xi,\eta) \geq 0$.
$(viii)$ $\tau_D := \inf\left\{t \geq 0, \left(X_1^{\xi}(t), y_1\left(X_1^{\xi}(.)\right)\right), \left(X_2^{\eta}(t), y_2\left(X_2^{\eta}(.)\right)\right) \notin D\right\} < \infty\ a.s$

$Q^{\xi\,\eta}\ \forall\ (\xi,\eta) \geq 0$.
Then, with
$y_1(\xi) = \int_{-\delta}^{0} \exp(\lambda s)\, \xi(s)\, ds$ and $y_2(\eta) = \int_{-\delta}^{0} \exp(\lambda s)\, \eta(s)\, ds$
and

$$\tau^* = \tau_D \qquad (3.6)$$

is an optimal stopping time for problem (2.1), for the proof see (10.4.1) in [Ø].

IV. EXPLICIT SOLUTION OF PROBLEM (2.1)- CONSTANTS COEFFICIENTS CASE

In equations (1.1) – (1.4) if the coefficients β_1 , σ_1 , β_2 and σ_2 are such that :

$$d X_1(t) = \left[\theta_1 X_1(t) + \alpha_1 \int_{-\delta}^{0} \exp(\lambda s)\left(X_1(t+s)\right) ds + \beta_1 X_1(t+\delta)\right] dt$$

$$+ X_1(t) + \beta_1 \exp(\lambda \delta)\left(\int_{-\delta}^{0} \exp(\lambda s)\left(X_1(t+s)\right) ds\right) \sigma_1\, dB_1(t) \ , t \geq 0 \qquad (4.1)$$

$$X_1(s) = \xi(s) \qquad for\ -\delta \leq 0 \leq 0 \qquad (4.2)$$

and

$$d\,X_1(t) = \Big[\ \theta_2\,X_2(t) + \alpha_2 \int_{-\delta}^{0} \exp(\lambda s)\left(X_2(t+s)\right)ds + \beta_2\,X_2(t+\delta)\Big]dt$$

$$+\,X_2(t) + \beta_2 \exp(\lambda\delta)\left(\int_{-\delta}^{0} \exp(\lambda s)\left(X_2(t+s)\right)ds\right)\sigma_2\,d\,B_2(t)\ ,\ t \ge 0 \qquad (4.3)$$

$$X_2(s) = \eta\ (s) \qquad for\ -\delta \le 0 \le 0 \qquad (4.4)$$

Then in view of theorem (1) we try to find a function φ of the form

$$\phi\left(x_1, x_2, y_1, y_2\right) = \psi\left(x_1, x_2, y_1, y_2\right)\ v_1(x_1, y_1) \le \mu v_2(x_2, y_2)$$

$$\phi\left(x_1, x_2, y_1, y_2\right) = g\left(x_1, x_2, y_1, y_2\right) v_1(x_1, y_1) \ge \mu\ v_2(x_2, y_2)$$

Where μ is a constant and $g\left(x_1, x_2, y_1, y_2\right) = x_1 - x_2 + \exp(\lambda\ \delta)\left(\beta_1\,y_1 - \beta_2\,y_2\right)$

The condition $L\ \varphi = 0$ gives

$$\left(\theta_1\,x_1 + \alpha_1\,y_1\right)\frac{\partial\psi}{\partial x_1} + \left(\theta_2\,x_2 + \alpha_2\,y_2\right)\frac{\partial\psi}{\partial x_2} + \frac{1}{2}\sigma_1^2\left(\theta_1\,x_1 + \beta_1 \exp(\lambda\delta)\ y_1\right)^2\frac{\partial^2\psi}{\partial x_1^2}$$

$$\frac{1}{2}\sigma_2^2\left(\theta_2\,x_2 + \beta_2 \exp(\lambda\delta)\ y_2\right)^2\frac{\partial^2\psi}{\partial x_2^2} + \left(x_1 - \lambda\,y_1\right)\frac{\partial\psi}{\partial y_1} + \left(x_2 - \lambda\,y_2\right)\frac{\partial\psi}{\partial y_2}$$

$$+z_1\left(\beta_1\frac{\partial\psi}{\partial x_1}\exp(-\lambda\ \delta)\frac{\partial\psi}{\partial y_1}\right) + z_2\left(\beta_2\frac{\partial\psi}{\partial x_2}\exp(-\lambda\ \delta)\frac{\partial\psi}{\partial y_2}\right) +$$

$$2\sigma_1\,\sigma_2\left(\theta_1\,x_1 + \beta_1\ \exp(\lambda\ \delta)\ y_1\right)\left(\theta_2\,x_2 + \beta_2 \exp(\lambda\delta)\ y_2\right)\frac{\partial^2\psi}{\partial x_1\,\partial x_2} = 0 \qquad (4.5)$$

Therefore $L\ \varphi = 0$ for all $z_1\ ,\ z_2$ if and only if

$$\beta_i\ \frac{\partial\psi}{\partial x_i} - \exp(-\lambda\ \delta)\frac{\partial\psi}{\partial y_i} = 0 \qquad ,\ i = 1,2 \qquad (4.6)$$

and

$$\left(\theta_1 x_1 + \alpha_1 y_1\right)\frac{\partial \psi}{\partial x_1} + \left(\theta_2 x_2 + \alpha_2 y_2\right)\frac{\partial \psi}{\partial x_2} + \frac{1}{2}\sigma_1^2\left(\theta_1 x_1 + \beta_1 \exp(\lambda\delta)\, y_1\right)^2 \frac{\partial^2 \psi}{\partial x_1^2}$$

$$\frac{1}{2}\sigma_2^2\left(\theta_2 x_2 + \beta_2 \exp(\lambda\delta)\, y_2\right)^2 \frac{\partial^2 \psi}{\partial x_2^2} + \left(x_1 - \lambda y_1\right)\frac{\partial \psi}{\partial y_1} + \left(x_2 - \lambda y_2\right)\frac{\partial \psi}{\partial y_2}$$

$$+ 2\sigma_1\sigma_2\left(\theta_1 x_1 + \beta_1 \exp(\lambda\delta)\, y_1\right)\left(\theta_2 x_2 + \beta_2 \exp(\lambda\delta)\, y_2\right)\frac{\partial^2 \psi}{\partial x_1 \partial x_2} = 0 \qquad (4.7)$$

Equation (4.7) holds if $\psi(x_1, x_2, y_1, y_2) = h(v_1, v_2)$ for some function $h : R^2 \to R$ where

$$v_1(x_1, y_1) = x_1 + \beta_1 \exp(\lambda\delta)\, y_1 \qquad (4.8)$$

$$v_2(x_2, y_2) = x_2 + \beta_2 \exp(\lambda\delta)\, y_2 \qquad (4.9)$$

Substituting this into (3.6) we get:

$$(\theta_1 + \beta_1 \exp(\lambda\delta)\left[x_1 + \left(\theta_1 + \beta_1 \exp(\lambda\delta)^{-1}\right)\left(\alpha_1 - \lambda\beta_1 \exp(\lambda\delta)\right)y_1 \right]\frac{\partial h}{\partial v_1}$$

$$+ (\theta_2 + \beta_2 \exp(\lambda\delta)\left[x_2 + \left(\theta_2 + \beta_2 \exp(\lambda\delta)^{-1}\right)\left(\alpha_1 - \lambda\beta_1 \exp(\lambda\delta)\right)y_2 \right]\frac{\partial h}{\partial v_2}$$

$$+\frac{1}{2}\sigma_1^2\, v_1^2 \frac{\partial h}{\partial v_1} + \frac{1}{z_2}\, v_2^2 \frac{\partial^2 h}{\partial v_2^2} + z_2\,\sigma_2\, v_1 v_2 \frac{\partial^2 h}{\partial v_1 v_2} = 0 \qquad (4.10)$$

Suppose that

$$\alpha_1 = \beta_1 \exp(\lambda\delta)\left(\lambda + \theta_1 + \beta_1 \exp(\lambda\delta)\right) \qquad (4.11)$$

$$\alpha_2 = \beta_2 \exp(\lambda\delta)\left(\lambda + \theta_2 + \beta_2 \exp(\lambda\delta)\right) \qquad (4.12)$$

Then (3.10) gets the form

$$\left(\theta_1 + \beta_1 \exp(\lambda\delta)\right)v_1 \frac{\partial h}{\partial v_1} + \left(\theta_2 + \beta_2 \exp(\lambda\delta)\right) v_2 \frac{\partial h}{\partial v_2} + \frac{1}{2}\sigma_1^2\, v_1^2 \frac{\partial h}{\partial v_1}$$

$$+ \frac{1}{z_2} v_2^2 \frac{\partial^2 h}{\partial v_2^2} + z_2\,\sigma_2\, v_1 v_2 \frac{\partial^2 h}{\partial v_1 \partial v_2} = 0 \qquad (4.13)$$

Let us try as candidate for solution

$$h(v_1, v_2) = C\, v_1^{\gamma}\, v_2^{1-\gamma} \qquad (4.14)$$

For suitable values the constants $C \geq 0$ and $\gamma \geq 0$. Equation (4.13) then gives

$$L\psi = C\, v_1^{\gamma}\, v_2^{1-\gamma}\left[\frac{1}{2}\gamma^2\, v\left(r_1 - r_2 - \frac{1}{2} v\right)\gamma + r_2\right] \qquad (4.15)$$

where

$$r_1 := \left(\theta_1 + \beta_1 \exp\ (\lambda\ \delta) \right) \tag{4.16}$$

$$r_2 := \left(\theta_2 + \beta_2 \exp\ (\lambda\ \delta) \right) \tag{4.17}$$

and

$$v = \sigma_1^2 - 2\ \sigma_1 \sigma_2\ \sigma_2^2 \tag{4.18}$$

Hence

$$L\ \psi = L\ h\,(v_1, v_2) = 0 \tag{4.19}$$

if and only if satisfies the equation

$$\frac{1}{2}\ \gamma^2\ v + \left(r_1 - r_2 - \frac{1}{2}\ v \right) \gamma + r_2 = 0 \tag{4.20}$$

The solutions of this equation are:

$$\gamma = \frac{1}{v} \left(\frac{1}{2}\ v + r_2 - r_1 \pm \sqrt{\left(r_1 - r_2 - \frac{1}{2} v \right)^2 - 2\ r_2\ v} \right) \tag{4.21}$$

Since we need to have $\gamma \geq 0$, we must require r_2 r_1. From now on we choose the plus sing in (4.21). For this value of γ put

$\psi\left(x_1, x_2, y_1, y_2\right) = h\left(v_1, v_2\right) = C\ v_1^{\gamma}\ v_2^{1-\gamma}$ for some constants C.

The requirement $\psi = g$ when $v_1 = \mu\ v_2$ gives

$$C(\mu\ v_2)^{\gamma}\ v_2^{1-\gamma} = \mu\ v_2 - v_2\ .\ \forall\ v_2 \geq 0 \text{ or } C\,\mu^{\gamma} = \mu - 1$$

The requirement $L\ \psi = L g$ when $v_1 = \mu\ v_2$ gives $C\left(\mu\ v_2\right)^{\gamma-1}\ v_2^{1-\gamma} = 1$

and $C\ (1-\gamma)(\mu\ v_2)^{\gamma}\ v_2^{-\gamma} = -1$ that is $C\ \gamma\,\mu^{\gamma-1} = 1$, and $C\ (1-\gamma)\,\mu^{\gamma} = -1$

The last three equation have the unique solution

$$\mu = \frac{\gamma}{\gamma - 1}\ ,\quad C = \frac{1}{\gamma} \left(\frac{\gamma}{\gamma - 1} \right)^{\gamma - 1} \quad L\ g\ (v_1, v_2) \leq 0 \text{ iff } v_1 \geq \mu\ v_2$$

The optimal strategy is to stop the first time the process,

$\left(X_1^{\xi}\ (t),\ y_1\left(X_t^{\xi}\ (.)\right),\ X_2^{\eta}\ (t),\ y_2\left(X_t^{\eta}\ (.)\right)\right)$ exits form the domain

$$D := \left\{ \left(x_1, x_2, y_1, y_2\right) \in [0,\infty), [0,\infty), [0,\infty), [0,\infty) : x_1 + \beta_1 \exp\,(\lambda\delta)\ y_1 \geq \mu\ v_2 \right\} \tag{4.22}$$

We need to check when v_1 $\mu\, v_2$. To this define

$$K\ (v_1, v_2) = \psi\ (v_1, v_2).g\ (v_1, v_2) = C\,v_1^{\gamma}\,v_2^{(1-\gamma)} - v_1 + v_2\ ,\ v_1, v_2 \geq 0$$

Then we have $K\ (v_1, v_2) = 0$, if $v_1 = \mu\ v_2$. Moreover

$$\frac{\partial K}{\partial v_1} = C \; \gamma \, v_1^{\gamma-1} \; v_2^{\gamma-1} \; - 1 = 0 \; , \; if \;\; and \;\; only \;\; if \;\; \frac{v_1}{v_2} = \mu \tag{4.23}$$

Since $\frac{\partial k}{\partial v_1} \to -1 \; , \; v_1 \to 0$, we must have $\frac{\partial k}{\partial v_1} < 0$ for all $v_1 < \mu \, v_2$ and

therefore $K(v_1, v_2) > 0$ for $v_1 < \mu \, v_2$.

Thus we conclude that $\varphi = \Phi$ and $\tau^* = \tau_{D.}$

Theorem 2 Suppose α_1 , α_2 , r_1 , r_2 , μ , C and γ satisfy the following conditions: $r_2 > r_1$

where $r_i := (\theta_i + \beta_i \exp(\lambda \delta))$, $i = 1, 2$ and

$$\alpha_i := \beta_i \exp(\lambda \delta) \left(\lambda + \theta_i + \beta_i \exp(\lambda \delta)\right) \tag{4.24}$$

$$\mu = \frac{v}{v-1} \quad , \quad C = \frac{1}{v}\left(\frac{v}{v-1}\right)^{1-v} \quad \text{Then:}$$

$$\Phi(\varsigma, \eta) = \varphi(x_1, x_2, y_1, y_2) = \psi(x_1, x_2, y_1, y_2); \; v_1(x_1, y_1) < \mu \, v_2(x_2, y_2)$$

$$\Phi(\varsigma, \eta) = \varphi(x_1, x_2, y_1, y_2) = g(x_1, x_2, y_1, y_2); \; v_1(x_1, y_1) \geq \mu \, v_2(x_2, y_2)$$

The optimal stopping time $\tau^*$is the first time the process

$\left(X_1^{\xi}(t), \; y_1\left(X_t^{\xi}(.)\right), \; X_2^{\eta}(t), \; y_2\left(X_t^{\eta}(.)\right)\right)$ exist form the domain D, where

$$D := \{(x_1, x_2, y_1, y_2) \in [0, \infty), [0, \infty), [0, \infty), [0, \infty) : x_1 + \beta_1 \exp(\lambda \delta) \, y_1 \geq \mu \, v_2\} \tag{4.25}$$

Remark: If we let the delay δ approaches to zero then equations (4.1), (4.3) become

$$d \, X_1(t) = (\theta_1 + \beta_1 \;) X_0(t) \; dt + \sigma_1 \; X_0(t) \; d \, B_1(t)$$

$$d \, X_2(t) = (\theta_2 + \beta_2 \;) X_0(t) \; dt + \sigma_2 \; X_0(t) \; d \, B_2(t)$$

The corresponding solution is

$\Phi(x_1, x_2) = C \, x_1^{\gamma} \, x_2^{1=\gamma}$ for $x_1 < \mu \, x_2$ will then be the limit of

$\Phi(\varsigma, \xi) = \Phi_0(\varsigma, \xi)$, $\delta \to 0^+$ as in [HØ].

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